Phys 410
Fall 2013
Lecture #11 Summary
8 October, 2013

We considered the motion of the Foucault pendulum. The demonstration showed that the pendulum moves in a fixed plane, as seen from an inertial reference frame. An inertial observer sees that the plane of oscillation is fixed and that the forces acting on the bob create no torque that will cause the plane of oscillation to change. However, in a rotating reference frame, the pendulum appears to move in a series of planes that rotate clockwise, as seen from above (in the northern hemisphere). The pendulum is made of a light wire of length L supporting a bob of mass m. The equation of motion of the bob as seen in the non-inertial frame is $m\ddot{\vec{r}} = \vec{F}_{net} + 2m\dot{\vec{r}} \times \vec{\Omega} + m(\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$, where the net force identified from an inertial reference frame is the vector sum of tension in the wire and gravity: $\vec{F}_{net} = \vec{T} + m\vec{g}_0$. This is the bare gravity force that points toward the center of the earth. Last time we saw that bare gravity can be combined with the centrifugal force and re-named effective gravity: $\vec{g} = \vec{g}_0 + \Omega^2 R \sin\theta\, \hat{\rho}$. We designate "up" or the +z-direction to be the direction away from $\vec{g}$, and y to be the "north" direction, and x to be the "east" direction. In this way, the angular velocity vector for the earth $\vec{\Omega}$ points in the y-z plane at an angle θ with respect to the "up" (z) direction.

The z-motion of the bob is fairly simple, essentially reducing to the statement that T ≅ mg. The tension in the horizontal xy-plane is $T_x = -mgx/L$, and $T_y = -mgy/L$. The Coriolis force is found from the cross product $2m\dot{\vec{r}} \times \vec{\Omega}$. We write $\dot{\vec{r}} = (\dot{x}, \dot{y}, \dot{z})$ and $\vec{\Omega} = (0, \Omega\sin\theta, \Omega\cos\theta)$. After carrying out the cross product and putting the results into the equation of motion, broken down into components, we get: $m\ddot{x} = -\frac{mgx}{L} + 0 + 2m(\dot{y}\Omega\cos\theta - \dot{z}\Omega\sin\theta)$, and $m\ddot{y} = -\frac{mgy}{L} + 0 - 2m\dot{x}\Omega\cos\theta$. We shall drop the $\dot{z}$ term in the x-equation because it is the product of two small velocities, define the constants $\omega_0^2 \equiv g/L$, and $\Omega_z \equiv \Omega\cos\theta$, to get two coupled equations of motion:

$$\ddot{x} - 2\dot{y}\Omega_z + \omega_0^2 x = 0$$

$$\ddot{y} + 2\dot{x}\Omega_z + \omega_0^2 y = 0$$

The first and third terms alone would give un-coupled simple harmonic motion in the xy-plane. The coupling terms look like a form of dissipation (of the form $F_{dis} = -bv$) but in fact they represent a coupling of energy from one direction of motion to the other. The energy in the oscillations sloshes back and forth between x and y.

These equations can be combined in a manner similar to the equations for motion of a charged particle in a magnetic field. Take the first equation plus "i" times the second equation, and define the new dependent complex variable $\eta(t) \equiv x(t) + iy(t)$ to get a single equation: $\ddot{\eta} + i2\dot{\eta}\Omega_z + \omega_0^2\eta = 0$. Trying a solution of the form $\eta(t) = e^{-i\alpha t}$, we get an auxiliary equation with solutions $\alpha = \Omega_z \pm \sqrt{\omega_0^2 + \omega\Omega_z^2}$. Using the fact that the pendulum oscillates many times compared to the rotation period of the Earth (i.e. $\omega_0 \gg \Omega_z$) we come to the solution $\eta(t) = e^{-i\Omega_z t}\left(C_1 e^{-i\omega_0 t} + C_2 e^{+i\omega_0 t}\right)$. To supply initial conditions, consider pulling the pendulum bob to a displacement A in the east (x) direction ($y = 0$) and release it from rest. In this case one finds $C_1 = C_2 = A/2$, and the solution is $\eta(t) = Ae^{-i\Omega_z t}\cos(\omega_0 t)$. Taking the real and imaginary parts to get the actual equations of motion in real space gives $x(t) = A\cos(\Omega_z t)\cos(\omega_0 t)$ and $y(t) = -A\sin(\Omega_z t)\cos(\omega_0 t)$. The pendulum swings back and forth on a short time scale, described by the factor of $\cos(\omega_0 t)$. On longer time scales, the plane of oscillation rotates, as described by the factors of $\cos(\Omega_z t)$ and $-\sin(\Omega_z t)$, with $\omega_0 \gg \Omega_z$. This slow rotation of the plane of oscillation occurs at a frequency that depends on your (co-)latitude on the Earth $\Omega_z \equiv \Omega\cos\theta$, where the rotation frequency of the Earth is $\Omega = 7 \times 10^{-5}$ Rads/s.

Changing gears, we talked about the calculus of variations. The calculus of variations is used to find extremum values of integral functionals. An example is a calculation of the shortest distance between two points in a plane. One can write the distance in terms of an integral over the path from the designated starting point $(x_1 y_1)$ to the designated end point $(x_2 y_2)$ as $L = \int_1^2 ds = \int_1^2 \sqrt{dx^2 + dy^2}$. If we (arbitrarily) treat the x coordinate as the independent variable we can write the integral as $L = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}\, dx$, where we have written $(dy/dx)^2$ as $(y')^2$. Our objective is to find the path $y(x)$ that minimizes this integral. This is a problem in the calculus of variations.

A second example is Fermat's principle. This is the problem of how light propagates from point 1 to point 2 through a variable dielectric medium characterized by an index of refraction that varies with position in a plane as $n(x, y)$. The light moves with variable speed $v = c/n(x, y)$. Fermat's principle says that light will take the path that minimizes the time to travel between the two points: $time(1 \to 2) = \frac{1}{c}\int_{x_1}^{x_2} n(x, y)\sqrt{1 + (y')^2}\, dx$. Again we need to find the path $y(x)$ that minimizes this integral. This is another problem in the calculus of variations.

The Euler-Lagrange equation is derived by assuming that there is an infinite family of "wrong" trajectories between points 1 and 2 parameterized by the function $\eta(x)$ and the constant α as $Y(x) = y(x) + \alpha\eta(x)$. The objective is to minimize the integral $S = \int_{x_1}^{x_2} f[y(x), y'(x), x]\, dx$, and this will be accomplished by taking $dS/d\alpha$ and setting it equal to zero. The result, after integrating by parts, is that the following expression must be satisfied for all points $x_1 \leq x \leq x_2$: $\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$, called the Euler-Lagrange equation.

Going back to the shortest-distance-in-a-plane problem, we see that the function f in this case is $f = \sqrt{1 + (y')^2}$. In this case f does not depend explicitly on$y$, hence we can write $\frac{\partial f}{\partial y'} = \frac{y\prime}{\sqrt{1+(y\prime)^2}} = C$, a constant. This can be reduced to $y'(x) = m$, another constant. Integrating both sides with respect to x, we find $y(x) = mx + b$, which is the famous equation for a straight line. The Fermat's principle problem can be solved in a similar way once the index of refraction distribution $n(x, y)$, and the end points, are specified.